Euler's Identity
$e^{i\pi} = -1$

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November 16, 2011

Not often can something so elegant and so mystifying as Euler's Identity be proved in just a few lines. In my opinion the simplicity of this proof only adds to its beauty.

To understand why $e^{i\pi} = -1$ it is necessary to first understand that any function $f(x)$ can be described as function of its own derivatives, evaluated at a point a in the domain of x. This description is called a infinite Taylor series expansion, and is written as the following:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... \tag{1}$$

A Maclaurin series is the particular case where the evaluation point $a = 0$.

$$f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + ...$$

Therefore Maclaurin series expansions of e^x is the following:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \tag{2}$$

Now consider a Maclaurin series expansion around e^{ix}

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} ... \\ &= 1 + ix + i^2\frac{x^2}{2!} + i^3\frac{x^3}{3!} + i^4\frac{x^4}{4!} + i^5\frac{x^5}{5!} ... \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} ... \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} ...) \end{aligned} \tag{3}$$

Now consider Maclaurin series expansions of $cos(x)$ and $sin(x)$.

$$\begin{aligned} cos(x) &= cos(0) + -sin(0)(x) + \frac{-cos(0)}{2!}x^2 + \frac{sin(0)}{3!}x^3 + ... \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... \end{aligned} \tag{4}$$

$$\begin{aligned} sin(x) &= sin(0) + cos(0)(x) + \frac{-sin(0)}{2!}x^2 + \frac{-cos(0)}{3!}x^3 + ... \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} .. \end{aligned} \tag{5}$$

Therefore, $e^{ix} = cos(x) + isin(x)$. Evaluating this equation at $x = \pi$ yields the original claim $e^{i\pi} = -1$.